DEVELOPING LOCAL INSTRUCTION THEORY FOR LEARNING PERMUTATION USING REALISTIC MATHEMATICS EDUCATION

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Abstract

Permutation is fundamental topic in combinatorics that support students' understanding of probability, statistics, and mathematical reasoning. However, many students experience difficulties in distinguishing permutation from combination and in applying counting principles in contextual situations. This study aims to develop a local instruction theory (LIT) for learning permutation based on the Realistic Mathematics Education (RME) approach. The research employed design research consisting of three phases: preliminary design, teaching experiment, and retrospective analysis. The participants were senior high school students in Indonesia. Data were collected through observations, student worksheet, interviews, and classroom documentation. The results indicate that the developed learning trajectory supports students in constructing permutation concepts gradually from informal strategies toward formal mathematical representation. The learning trajectory includes contextual exploration, modelling using diagrams and tables, pattern recognition, generalization, and application of permutation formulas. The findings suggest that the developed LIT can facilitate students' conceptual understanding and mathematical modelling abilities. This study contributes to mathematics education research by providing a theoretically grounded instructional design for teaching permutation.

Keywords: permutation, local instruction theory, design research, realistic mathematics education

Abstrak

Permutasi merupakan salah satu topik penting dalam kombinatorika yang mendukung pemahaman siswa terhadap peluang, statistika, dan penalaran matematis. Namun, banyak siswa masih mengalami kesulitan dalam membedakan konsep permutasi dan kombinasi serta menerapkan prinsip pencacahan pada situasi kontekstual. Penelitian ini bertujuan mengembangkan teori instruksi lokal (Local Instruction Theory/LIT) pada pembelajaran permutasi berbasis pendekatan Realistic Mathematics Education (RME). Penelitian menggunakan metode design research yang meliputi tiga tahap, yaitu preliminary design, teaching experiment, dan retrospective analysis. Partisipan penelitian adalah siswa sekolah menengah atas di Indonesia. Data penelitian dikumpulkan melalui observasi, lembar kerja siswa, wawancara, dan dokumentasi pembelajaran. Hasil penelitian menunjukkan bahwa lintasan belajar yang dikembangkan mampu membantu siswa membangun konsep permutasi

secara bertahap dari strategi informal menuju representasi matematis formal. Lintasan belajar meliputi eksplorasi masalah kontekstual, pemodelan menggunakan diagram dan tabel, pengenalan pola, generalisasi, hingga penggunaan rumus permutasi. Selain itu, siswa menunjukkan peningkatan kemampuan dalam memahami konsep dan memodelkan masalah matematika. Temuan ini menunjukkan bahwa LIT berbasis RME dapat menjadi alternatif desain pembelajaran yang efektif untuk mendukung pemahaman konseptual siswa pada materi permutasi.
***Kata Kunci**: permutasi, teori instruksi lokal, penelitian desain, dan Pendidikan Matematika Realistik (PMR).*

1. Introduction

Combinatorics constitutes a fundamental domain in discrete mathematics, essential for cultivating students' logical reasoning, problem-solving capabilities, and higher-order thinking skills. Current global trends in mathematics education research strongly emphasize the necessity of designing learning environments that facilitate meaningful conceptual understanding rather than merely drilling procedural fluency (Bakker et al., 2021). However, empirical reality demonstrates that combinatorics, particularly permutation, remains a substantial cognitive challenge for students. A study conducted by (Lamanna et al., 2022) confirms that secondary school students frequently exhibit inconsistent strategies and tend to rely heavily on memorized formulas without profound conceptual underpinning when solving permutation and combination problems, even following formal instruction.

The primary root of these conceptual difficulties is closely intertwined with students' limited mathematical modelling competencies when confronting varied problem structures. The cognitive complexity inherent in learning permutation becomes increasingly evident when examining its diverse sub-topics: Each sub-topic necessitates a specific mathematization process. For permutations with distinct elements, students are required to comprehend the foundation of linear ordered arrangements. When transitioning to permutations with identical elements, the cognitive demand escalates significantly, as students must be capable of detecting and reducing counting redundancies (overcounting) caused by identical elements. Meanwhile, cyclic permutations introduce a spatial paradigm shift from linear to circular arrangements, compelling students to grasp the abstract concept of fixing a reference point. Traditional instructional practices frequently fail to bridge the modifications without facilitating logical meaning construction (Leung et al., 2021;Meika et al., 2019b)

To address this instructional gap, the Realistic Mathematics education (RME) approach provides a pedagogically sound and theoretically proven framework. Various recent literature reviews have affirmed the potential of RME in designing meaningful combinatorics instruction (Marina Angraini, 2020;Fitriawan et al., 2023 Somakim et al., 2018). This approach enables students to experience the "guided reinvention" of mathematical concepts through the modelling of contextual problems. For instance, the integration of empirical contexts (Ryandi et al., 2018) and experience-based activities such as role-playing have been shown to effectively stimulate students' informal strategies and facilitate their transition toward systematic combinatorial reasoning (Nopriana & Dwi Rosita, 2018; Somakim et al., 2019; Cahyadi & Noto, 2023).

Although the efficacy of the RME approach in combinatorics education has been well-documented, a review of the state-of-the-art reveals a critical research gap. The majority of previous studies concerning Local Instruction Theory (LIT) have predominantly focused on basic multiplication rule (Ningtyas et al., 2023) or have been exclusively tailored to combinations (Sari, 2017; Meika et al., 2019a). to date, there remains a lack of comprehensive studies specifically designed to develop and validate an LIT that facilitates an integrated learning trajectory encompassing permutation and all three of its sub-topics holistically. To fill this research void, this study aims to develop a valid and practical Local Instructional Theory for learning permutation (including distinct elements, identical elements, and syclic permutations) base on the Realistic Mathematics Education approach, ultimately guiding students' transition from situational comprehension to robust formal mathematical understanding.

2. Metode

This study employed design research methodology which focuses on developing and refining instructional designs through iterative cycles of design, implementation, and analysis. Freudenthal (1991) mentioned that cyclic process in design research consists of idea/ thought experiment and instruction experiment. The research consisted of three phases (K. Gravemeijer & Cobb, 2006): (1) Preliminary Design, literature review, curriculum analysis, and analysis of students' difficulties before designing the learning activities; (2) Teaching Experiment. This stage aims to collect data to be used to answer the research questions. A series of instructional activities that have been designed, tested and revised, are applied in the classroom. Research subjects in the teaching experiment were 19 students from the eleventh Natural Science-Major graders at SMAN 1 Pandeglang, Banten; (3) Retrospective Analysis, the researcher analyzed the data obtained of the teaching experiment and used the results of the analysis to develop the next design. Comparison between predicted and actual learning processes.

3. Result and Discussion

The result presented include the stage of preliminary design, teaching experiment, and retrospective analysis.

3.1 Preliminary design

In preliminary design, preparation and completion of teaching materials were done. Activities undertaken at the preparatory stage: (1) analyzed the students by determining the research subjects of eleventh Natural Science-major at SMAN 1 Pandeglang with cognitive level of heterogeneous students, (2) analyzed the curriculum to find out that the permutation material, and (3) analyzed the material to know that basic competence "describe and apply various combinations through diagrams or other means" is appropriate to the research objective.

In the perspective of design research, the purpose of the initial design is to formulate LIT that can be described and perfected during teaching experiment (Gravemeijer & Cobb, 2006). During this literature study, we also began designing learning activities. The sequence of this learning activity includes the conjectures of students' thought and strategies developed and presented as the initial HLT. The HLT thought is dynamic and can be changed and adapted to the actual student learning process during the teaching experiment. HLT in teaching experiments is described in Tabel 1.

Tabel 1. Hypothetical Learning Trajectory for Learning Permutation

Activity	Learning Goals	Description of Learning Activity	Conjectures of Students' Thinking	Teacher's Anticipation/ Intervention
Permutation of distinct elements				
Activity 1: Arranging distinct letters (experiment)	Students understand that order matters in permutations	Teacher: provides envelopes containing distinct letters; poses a problem: "How many different arrangements can you form?"; encourages students to record all possibilities. Students: arrange letters physically; list	Some students may recognize that different orders produce different outcomes (AB ≠ BA). Some may omit or repeat cases due to unsystematic listing.	Teacher introduces organizing tools (tree diagram/table); prompts students to check completeness; highlights importance of order.

Activity	Learning Goals	Description of Learning Activity	Conjectures of Students' Thinking	Teacher's Anticipation/ Intervention
		all possible permutations; compare results within group; presents their findings.		
Activity 2: Selecting committee members (contextual problem)	Students understand permutation of r elements from n elements.	Teacher: Presents contextual problem (e.g., selecting a president and vice president); asks guiding questions about roles and positions; encourages use of prior models. Students: Model the problem using diagrams/tables; calculate number of possible selections; discuss why position matters; attempt to generalize patterns.	Some students may realize that roles are position-sensitive. Students begin identifying relationships between n, r, and total arrangements.	Teacher guides students toward generalization using factorial relationships; facilitates transition from informal models to formal expression.
Permutation of identical elements				
Activity 3: Arranging letters with identical elements (experiment)	Students recognize overcounting due to identical elements	Teacher: Provides letter sets with repetitions; ask students to determine total arrangements; encourages comparison between groups. Students: Arrange letters;	Some students initially count all permutations without considering duplicates. Some may gradually recognize repetition.	Teacher creat cognitive conflict by comparing results; prompts identification of identical arrangements; introduces idea of eliminating duplicates

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Activity	Learning Goals	Description of Learning Activity	Conjectures of Students' Thinking	Teacher's Anticipation/ Intervention
		list permutations; notice some arrangements appear identical; begin questioning results.		
Activity 4: Contextual problem with identical objects	Students derive the formula for permutation with identical elements.	Teacher: presents real-life problem (e.g., arranging identical books); ask students to connect with previous findings; guides pattern recognition. Students: apply previous strategies; group identical outcomes; attempt to adjust counting method; construct formula using factorial division.	Some students may begin dividing by repeated factorials; construct formula $\frac{n!}{k_1!k_2!}$.	Teacher supports formalization; validates formula through substitution and verification; emphasizes meaning of each parameter.
Cyclic permutation				
Activity 5: Circular seating demonstration (3 students)	Students understand rotational equivalence	Teacher: Invites students to stand in a circle; instructs them to change positions; asks class to observe differences; introduces the idea of a reference point. Students: Perform circular	Students initially consider all arrangements different; begin to realize some are equivalent due to rotation.	Teacher introduces "fixed reference point" (center); guides students to identify equivalent arrangements.

Activity	Learning Goals	Description of Learning Activity	Conjectures of Students' Thinking	Teacher's Anticipation/ Intervention
		arrangements; record observed patterns; compare different configurations.		
Activity 6: Exploring 4 elements in a circle	Students identify patterns in cyclic permutations	Teacher: Assigns task to determine number of distinct arrangements for 4 objects in circle; encourages modeling through sketches. Students: Draw circular arrangements; list possibilities; identify patterns; eliminate equivalent cases.	Students begin systematic enumeration but may struggle distinguishing equivalent rotations.	Teacher scaffolds identification of duplicates patterns; reinforces structural reasoning.
Activity 7: Generalizing circular permutation formula	Students derive the formula for cyclic permutations	Teacher: Prompts students to analyze result from previous activities; asks guiding questions about patterns and relationships. Students: Compare results for different values of n; identify pattern; formulate general rule $(n-1)!$; justify reasoning.	Students recognize that fixing one element reduces permutations; derive $(n-1)!$.	Teacher formalized reasoning; connects with factorial concept; validates generalization through examples.

3.2 Teaching experiment

According to Gravemeijer & Cobb (2006), the teaching experiment phase was conducted to implement the designed HLT in a real classroom setting and to examine its validity. The learning process was carried out using the RME approach, emphasizing contextual activities, group discussions, and mathematization processes.

1) Implementation of permutations of distinct elements (2 activities)

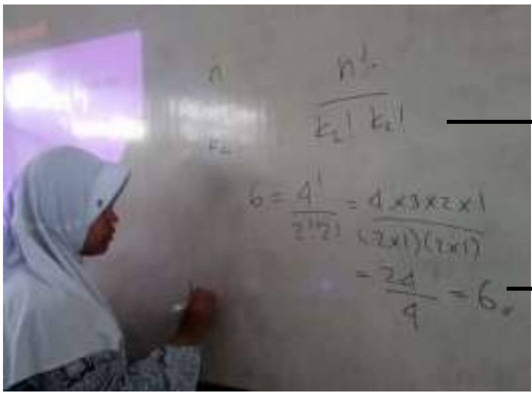
   In the first activity, the teacher facilitated students in arranging distinct letters provided in envelopes. Students listed all possible arrangements and discussed their results in groups. Various representations such as systematic lists and tree diagrams emerged as model of.

   In the second activity, the teacher presented a contextual problem involving role selection. Students applied their previous models to determine the number of possible arrangements. Classroom discussion indicated that students began to understand the importance of order and connected their findings to factorial structures, leading to vertical mathematization.

2) Implementation of permutations of identical elements (2 activities)

   In the first activity, students arranged letters with repeated elements. The results showed a tendency toward overcounting, as students initially counted all arrangements without recognizing identical cases. Through teacher scaffolding and discussion, students began identifying duplicates.

   In second activity, the teacher introduced contextual problems involving identical objects. Students linked their prior findings with factorial concepts and gradually constructed the formal formula by dividing repeated arrangements.



Formal formula: the result of vertical

Formal formula reconstruction

Figure 1. Process of Reconstruction of the Permutation Formula of the Same Elements

3) Implementation of cyclic permutation (3 activities)

In the first activity, students participated in a circular arrangement demonstration. They observed and recorded patterns, then discussed equivalence due to rotation.

In the second activity, students explore arrangements of four elements in a circular setting using sketches and pattern listing. Some difficulties emerged in identifying equivalent rotations.

Figure 2. Circular Position Swap Demonstration in Cyclic Permutation Learning

In the third activity, students generalized their findings and concluded that the number of cyclic permutation is $(n-1)!$, understanding the concept of fixing one element.

Overall, the implementation demonstrated that the HLT was applicable, although variations in student strategies and understanding were observed.

3.3 Retrospective analysis

The retrospective analysis compare the designed HLT with empirical data from the teaching experiment to evaluate its effectiveness and identify areas for improvement. Specifically, this analysis examine the efficacy of the LIT instruments, grounded in the RME approach, in facilitating students' conceptual understanding and combinatorial reasoning across three permutation sub topics: permutations with distinct elements, permutations with identical elements, and cyclic permutations.

1) Learning trajectory of permutations with distinct elements.

During the letter arrangement activities and the student council election simulation, students were presented with contextual problems designed to establish the fundamental understanding that the order of arrangements holds distinct computational significance ($AB \neq BA$). Observations from teaching experiment demonstrated that the utilization of self-developed models, such as tree diagrams and grid representations, was highly effective in guiding students' horizontal mathematization process. In the vertical mathematization stage, discursive interactions by a student group presentation, demonstrated strong cognitive abilities in reducing informal computational tables into formal factorial representations. The students logically proved that 6 variations from 3 candidates for 2 positions could be formulated through the factorial division operation $\frac{3!}{(3-2)!}$. This exploration independently guided them to discover the formal permutation formula $P_r^n = \frac{n!}{(n-r)!}$. this process represents an optimal application of guded reinvention, supporting (K. P. E. Gravemeijer et al., 2012) assertion that guided rediscovery enables students to construct profound and meaningful mathematical conceptual foundations. The LIT flow of distinct element permutations before and after learning is presented in Figure 3

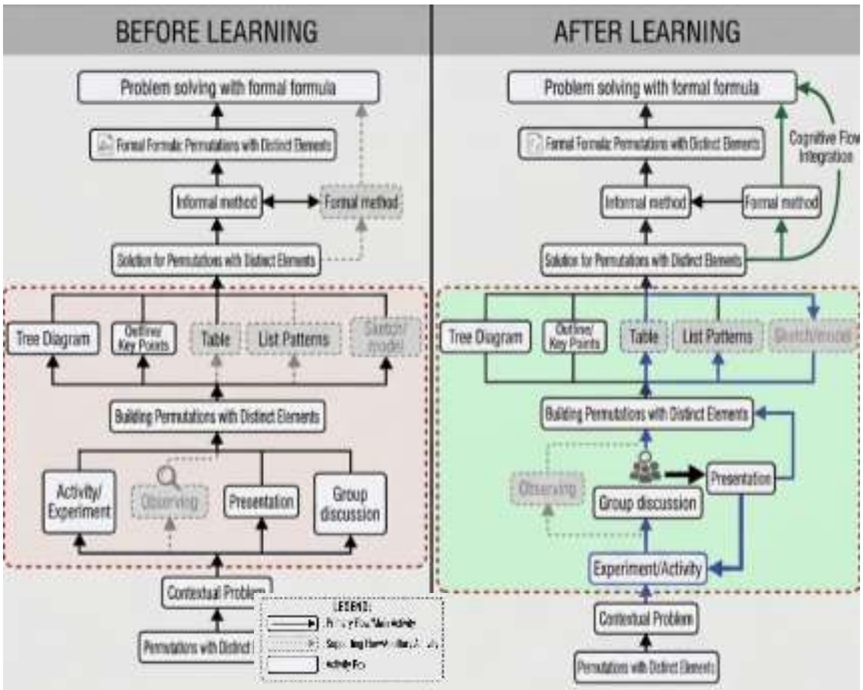


Figure 3. LIT flow of distinct element permutations

2) Learning trajectory of permutations with identical elements

The cognitive complexity in permutations with identical elements lies in the epistemological challenge of identifying and reducing counting redundancies (overcounting) caused by the presence of identical elements. In the password arrangement activity, students initially employed informal strategies by listing letters arrangement patterns. An essential finding in this phase was the success of a student group (see Figure 1) in analytically reconstructing the formula. Through teacher scaffolding, students realized that the total number of arrangements $(n!)$ to eliminate irrelevant duplicate arrangements. The students' success in formulating the algorithm $\frac{n!}{k_1!k_2!}$ and empirically proving it for the case of $n = 4$ with $k_1 = 2$, $k_2 = 2$ proves that the presented contextual didactical phenomena can comprehensively bridge students' combinatorial understanding structures (Batanero et al., 1997). The LIT flow of permutations with identical elements is the same as the flow in learning of permutation with distinct elements.

3) Learning trajectory of cyclic permutations

Cyclic permutations demand a spatial paradigm shift from a linear orientation to a circular arrangement. Responding to the visual confusion observed during teaching experiment. Implementation was strategically modified using physical demonstrations (role-playing) where three students stood in a circle. The retrospective analysis revealed that the instruction to designate one individual as a "fixed reference point" (centre) was a crucial cognitive turning point for students' comprehension (see Figure 2). By establishing this reference point, students could detect and distinguish that the arrangement ABC, BCA, and CAB constituted a single equivalent cyclic position rather than three distinct linear positions. This spatial representation successfully facilitated students in abstracting that 3 elements produce 2 distinct cyclic patterns, and 4 objects produce 6 distinct patterns, culminating in the generalization of the final formulas $(n-1)!$. The success of this pedagogical intervention supports (Lockwood, 2013) theory that structured outcome-listing activities are a primary key to overcoming ambiguities in combinatorial mastery. The LIT flow of cyclical permutations before and after learning is presented in Figure 4.

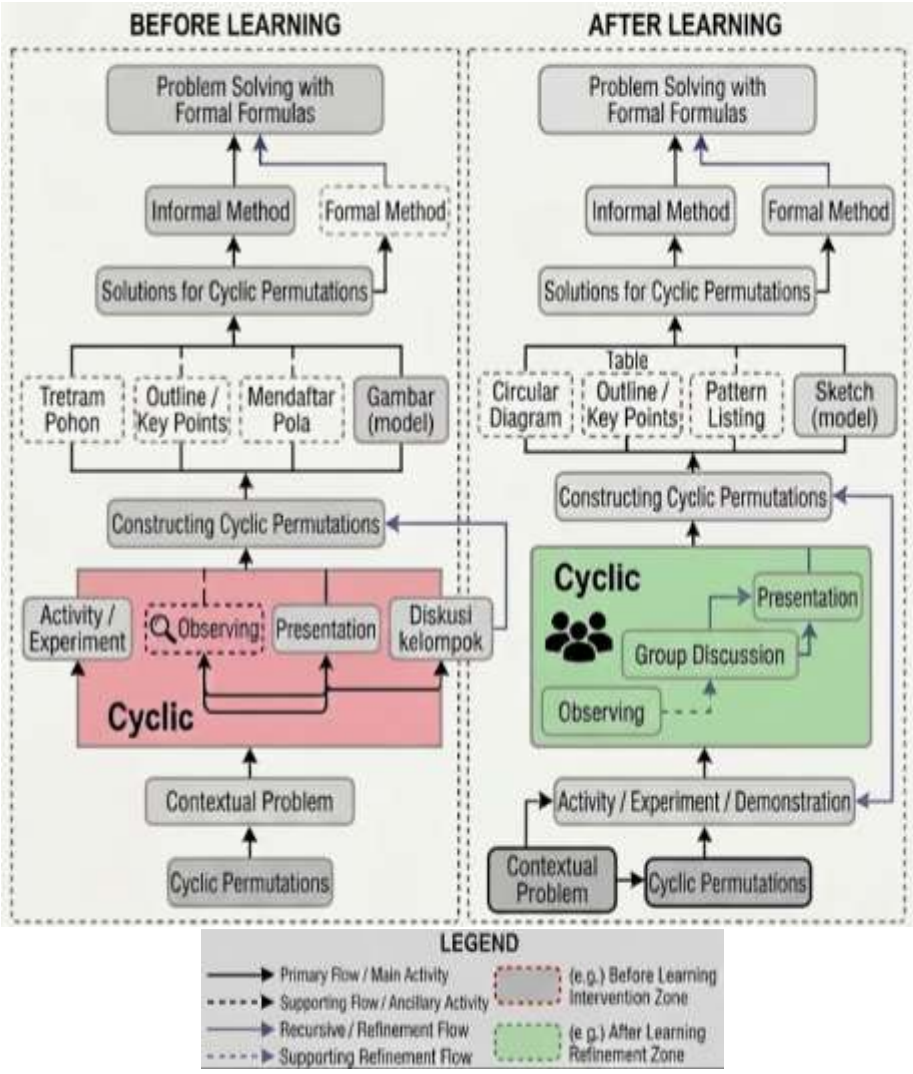


Figure 4. LIT of cyclical permutations before and after learning

Conclusion

Drawing upon the retrospective analysis and teaching experiment outcomes, this study concludes that the Local Instruction Theory (LIT) based on Realistic Mathematics Education (RME) developed for learning permutations is valid, practical, and effective. The designed learning trajectory significantly facilitates students' cognitive transition from horizontal mathematization to vertical mathematization, culminating in the formal construction of formulas for permutations with distinct elements, identical elements, and cyclic permutations. Specifically, the principle of guided reinvention successfully directs students in reducing empirical situations, detecting counting redundancies, and resolving spatial disorientation by establishing a fixed reference point. This demonstrates that the RME approach effectively transforms combinatorics education from the transmission of ready-made mathematics into an analytical, independent meaning-making activity. Theoretically and practically, the implications of this research enrich the mathematics education literature by establishing that epistemological obstacle in complex topic can be dismantled through the presentation of contextual didactical phenomena, thereby redefining the educator's role as a facilitator of knowledge construction. Consequently, it is recommended that mathematics educators adopt this learning trajectory in regular instructional setting to minimize students' misconceptions, while future studies should integrate this instructional theory with dynamic mathematics software and broaden the demographic scope of testing to substantiates its external validity.

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